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1992 J. Phys. A: Math. Gen. 25 L837

(http://iopscience.iop.org/0305-4470/25/13/012)

J. Phys. A: Math. Gen. 25 (1992) L837–L842. Printed in the UK

LETTER TO THE EDITOR

Classical spin dynamics and quantum algebras

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Received 14 October 1991

Abstract. It is shown that the classical dynamics of a charged spinning particle in a constant homogeneous magnetic field is compatible with an $SU_q(2)$ Poisson bracket structure for the generators of spin angular momentum. The novel feature of our analysis is that it is not based on a Hamiltonian formulation of the equations of motion.

Attention has been drawn recently to the quantal phase of Berry (1984, 1985) accompanying adiabatic changes and its classical analogue called Hannay angles (Hannay 1985). Berry and Hannay have established this phenomenon in full generality and also explicitly worked out several interesting cases. In particular, Berry has calculated the geometrical phase factor for an arbitrary spin eigenstate when the external magnetic field about which the spin magnetic moment precesses is varied adiabatically around a closed circuit. Many authors have reported measurements of the Berry phase of a spinning particle in a magnetic field. The results of these measurements agree with classical phase shifts calculated by solving the equations of spin precession (see e.g. Cina 1986). In this work we point out that these classical equations admit a Poisson bracket structure which is more general than the conventional SU(2) Lie algebraic structure assumed for the spin variables.

Consider the classical description of a non-relativistic particle with definite spin. Assume that the particle has an electric charge q, a mass m and a magnetic moment μ. If the particle also interacts with an external magnetic field B_a which does not explicitly involve time, the familiar equations of motion are

$$m\ddot{x}_a = q\varepsilon_{abc}\dot{x}_b B_c + \mu S_b\, \partial B_b/\partial x_a \tag{1}$$

$$\dot{S}_a = \mu\varepsilon_{abc} S_b B_c. \tag{2}$$

Here x_a and S_a denote, respectively, the position and the spin degrees of freedom. As is well known, the ponderomotive equation (1) describes the motion of a charged spinning particle in a non-homogeneous magnetic field. Of importance for calculating classical phase shifts is (2) which describes the spin magnetic moment precession in a magnetic field. The equations of magnetic field are the Maxwell equations.

The classical Hamiltonian description of (1) and (2) is known. In the standard theory the Hamiltonian has the following form (where A_a is the vector potential from which the magnetic field is derived in the usual way, $B_a = e_{abc}\, \partial A_c/\partial x_b$, in order to make B_a a solenoidal field)

$$H = (p_a - qA_a)(p_a - qA_a)/2m - \mu S_a B_a. \tag{3}$$

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0305-4470/92/130837+06$04.50

The various Poisson brackets are

$$\{x_a, x_b\} = 0 \tag{4}$$

$$\{p_a, p_b\} = 0 \tag{5}$$

$$\{x_a, p_b\} = \delta_{ab} \tag{6}$$

together with

$$\{S_a, S_b\} = \varepsilon_{abc} S_c \tag{7}$$

and

$$\{S_a, x_b\} = 0 \tag{8}$$

$$\{S_a, p_b\} = 0. \tag{9}$$

The Poisson brackets given in (7) are a consequence of the assumption which is conventionally made that S_a obey the SU(2) algebra.

Now it is easy to check using (3)–(9) that one can derive (1) and (2) from

$$\dot{p}_a = \{p_a, H\} = -\partial H/\partial x_a \tag{10}$$

$$\dot{x}_a = \{x_a, H\} = +\partial H/\partial p_a \tag{11}$$

$$\dot{S}_a = \{S_a, H\} = \varepsilon_{abc} S_c \, \partial H/\partial S_b. \tag{12}$$

Note that H does not explicitly depend on time and is a conserved quantity by virtue of (10)–(12).

It is worth pointing out that (1) and (2) also admit a classical description outside the framework of conventional wisdom just described. This conceptually simpler description is based on the following Poisson bracket relations (all of which are compatible with the aforementioned ones)

$$\{x_a, \dot{x}_b\} = \frac{1}{m} \delta_{ab} \tag{13}$$

$$\{\dot{x}_a, \dot{x}_b\} = \frac{q}{m^2} \varepsilon_{abc} B_c \tag{14}$$

besides

$$\{S_a, \dot{x}_b\} = 0 \tag{15}$$

the relation (8), and the SU(2) relations assumed in (7). The relation (14) may be said to define the quantity B_a. Following Dyson (1990) it can be shown that B_a is velocity independent and solenoidal. The coefficient q/m^2 will correctly lead to the 'q against cross B' term in the Lorentz force (see (22)). In this way the relation (14) should not be regarded as an additional assumption; rather it may be looked upon as the definition of magnetic field. The simplicity of our approach lies in essentially assuming (13), leaving aside the familiar assumptions of (7), (8) and (15).

To arrive at the equations of motion we may proceed as follows. Since (13), (14), (15), (8) and (7) are valid at all times, we are led to the following consistency conditions which are obtained by differentiating them. We have, respectively,

$$\{\dot{x}_a, \dot{x}_b\} + \{x_a, \ddot{x}_b\} = 0 \tag{16}$$

$$\{\ddot{x}_a, \dot{x}_b\} + \{\dot{x}_a, \ddot{x}_b\} = \frac{q}{m^2} \varepsilon_{abc}\dot{B}_c \tag{17}$$

$$\{\dot{S}_a, \dot{x}_b\} + \{S_a, \ddot{x}_b\} = 0 \tag{18}$$

$$\{\dot{S}_a, x_b\} + \{S_a, \dot{x}_b\} = 0 \tag{19}$$

$$\{\dot{S}_a, S_b\} - \{\dot{S}_b, S_a\} = \varepsilon_{abc}S_c. \tag{20}$$

These conditions are 'weak' in the sense that they hold by virtue of the basic bracket relations. They lead to the following 'strong' conditions, however.

Consider (16). It leads to

$$m\, \partial\ddot{x}_a/\partial\dot{x}_b = q\varepsilon_{abc}B_c \tag{21}$$

whereupon its solution gives

$$m\ddot{x}_a = q\varepsilon_{abc}\dot{x}_b B_c + E_a \tag{22}$$

in which the newly introduced quantity E_a denotes a velocity independent field, though it may depend on position and spin variables. We may substitute (22) into (17) to discover that E_a is irrotational: $\varepsilon_{abc}\, \partial E_c/\partial x_b = 0$. Hence we may write

$$m\ddot{x}_a = q\varepsilon_{abc}\dot{x}_b B_c - \partial V/\partial x_a \tag{23}$$

where V is the velocity-independent potential energy function (the specific form for V will be chosen later)

$$\{V, x_a\} = 0. \tag{24}$$

Having established Newton's law of motion in the form given by (23) we may insert this law into (18) to yield

$$\partial\dot{S}_a/\partial x_b = \partial\{S_a, V\}/\partial x_b. \tag{25}$$

This last equation is easily solved to give the equation of motion for spin variables in the form

$$\dot{S}_a = \{S_a, V\} + X_a. \tag{26}$$

In order to be led to spin precession from (26) it is crucial to realize that the first term in the equation of motion for S_a, viz, $\{S_a, V\} = \{S_a, S_b\}\partial V/\partial S_b = \varepsilon_{abc}S_c\, \partial V/\partial S_b$, automatically satisfies the consistency condition given in (20), so X_a is bound to satisfy this condition separately

$$\{X_a, S_b\} - \{X_b, S_a\} = \varepsilon_{abc}X_c. \tag{27}$$

In turn this leaves no room for addition of X_a in (26). Let us argue that X_a must vanish. By definition X_a is independent of x_b. Now consider its bracket with x_b. Using the Jacobi identity involving V, x_b and S_a, we get

$$\{X_a, x_b\} = \{\dot{S}_a, x_b\} - \{\{V, x_b\}, S_a\} - \{\{x_b, S_a\}, V\}. \tag{28}$$

This bracket vanishes term by term in virtue of (15), (19), (8) and (24). Hence X_a must also be velocity independent and may only be of the form $X_a = S_a f(S_b S_b)$ and satisfy

$$\{S_a, X_b\} = \varepsilon_{abc}X_c. \tag{29}$$

Now (27) and (29) agree only if $X_a = 0$. In conclusion, the equation of motion for S_a assumes the form

$$\dot{S}_a = \{S_a, V\}. \tag{30}$$

We have shown the existence of a quantity $V(x, S)$ in terms of which our strong conditions are (23) and (30).

Let us complete the derivation of (1) and (2) which follow from (23) and (30). It is easy to check that $V = -\mu S_a B_a$ leads to (1) and (2). However we are free to add an arbitrary function of $C_1 = S_a S_a$ to V without affecting the equations of motion. Before we take issue with the deformed Poisson bracket structure in the following, let us agree to write the SU(2) Poisson bracket relations as

$$\{S_a, S_b\} = \tfrac{1}{2}\varepsilon_{abc}\, \partial C/\partial S_c \tag{31}$$

with $C = C_1$. With the help of (30) we get

$$\dot{C} = \dot{S}_a\, \partial C/\partial S_a = \{S_a, S_b\}\partial V/\partial S_b\, \partial C/\partial S_a \tag{32}$$

which vanishes on substituting (31) into (32). Hence C is constrained to be a constant of motion.

Having knit a framework suitable for the description of spin dynamics which is simpler than the existing descriptions (see e.g. Balachandran *et al* 1983), we are ready to establish our main claim in this work. Consider the interesting special case of a constant and homogeneous magnetic field. Let us assume that a magnetic field of given strength $B = (B_a B_a)^{1/2}$ points in the z-direction everywhere. In this case the equations of spin precession reduce to

$$\dot{S}_a = \mu B \varepsilon_{ab3} S_b \tag{33}$$

thereby showing that C_1 and $C_2 = S_3$ are both constants of motion. Furthermore when one tries to derive (33) from (30) with the choice $V = -\mu B S_3$, it is found that while $\{S_3, S_1\}$ and $\{S_3, S_2\}$ are uniquely determined, $\{S_1, S_2\}$ is allowed to be an arbitrary function of S_3. It is legitimate to ask whether one may depart from the SU(2) Lie algebraic Poisson bracket structure by modifying the definition of C given in (31). We are now going to show that our analysis leading to the equations of spin precession (33) remains unchanged under the modification of C produced by the addition of an arbitrary function of C_2^2 to C_1 in the definition of C.

Let us say that the Poisson brackets are defined as in (31) with $C = C_1 + g(C_2^2)$. Here g is an arbitrary function of its argument. The modified Poisson bracket relations for S_a become

$$\{S_a, S_b\} = \varepsilon_{abc} S_c + \varepsilon_{ab3} S_3\, \mathrm{d}g/\mathrm{d}C_2^2. \tag{34}$$

We can recover the consistency condition given in (20) by taking $\mathrm{d}/\mathrm{d}t$ of (34), so this condition is unaffected under the modification of C given above. This condition ultimately led to (30). The crucial point in our argument leading to (30) from (26) was the identity

$$\{S_a, \{S_b, V\}\} - \{S_b, \{S_a, V\}\} = \varepsilon_{abc}\{S_c, V\}. \tag{35}$$

This last result (35) holds just as well in the present case. This is essentially a consequence of the Jacobi identity involving S_a, S_b and S_3. This identity is trivial to check with the help of (34) because (34) does not modify $\{S_3, S_1\}$ and $\{S_3, S_2\}$. In balance we have shown that the equations (33) follow from (30) with the help of also the modified Poisson bracket relations as given in (34). We need to take $V=-\mu BS_3$ up to the addition of an arbitrary function of C given above.

In order to obtain an $SU_q(2)$ algebraic structure we may choose C to be the Casimir invariant for $SU_q(2)$, i.e.

$$C=C_q=C_1-C_2^2+\frac{\sinh n}{n}\left(\frac{\sinh nC_2}{\sinh n}\right)^2. \tag{36}$$

Here n is related to the deformation parameter q as $n=\ln q$. Together with equation (31), this definition of C yields

$$\{S_1, S_2\}=\frac{1}{2}\frac{\sinh(2nS_3)}{\sinh n} \qquad \{S_1, S_3\}=-S_2 \qquad \{S_2, S_3\}=S_1. \tag{37}$$

This is the well known deformation of the universal enveloping algebra of SU(2) (Jimbo 1986).

It is worth pointing out that we can be even more general and describe the equations of spin precession (33) by means of a pair of functions depending arbitrarily on C_1 and C_2. Let the fundamental brackets be defined by (Hojman 1991)

$$\{S_a, S_b\}=l\varepsilon_{abc}\,\partial C/\partial S_c \tag{38}$$

for any functions l and C of C_1 and C_2. It is important to observe that this definition ensures the antisymmetry property $\{S_a, S_b\}=-\{S_b, S_a\}$ and the Jacobi identity $\varepsilon_{abc}\{\{S_a, S_b\}, S_c\}=0$. The constraint equations

$$\{S_a, \dot S_b\}-\{S_b, \dot S_a\}=l\varepsilon_{abe}\dot S_d\frac{\partial^2 C}{\partial S_e\,\partial S_d} \tag{39}$$

obtained by differentiating (38) are automatically satisfied by the equations of motion in the form

$$\dot S_a=\{S_a, V\} \tag{40}$$

by virtue of the Jacobi identity involving S_a, S_b and S_c. Here V is any function of C_1 and C_2. It is easy to verify that the equations of spin precession (33) follow from (40) provided we choose

$$l=-\mu B/2\left(\frac{\partial C}{\partial C_1}\frac{\partial V}{\partial C_2}-\frac{\partial C}{\partial C_2}\frac{\partial V}{\partial C_1}\right)^{-1}. \tag{41}$$

The choice $l=\frac{1}{2}$, $C=C_q$, $V=-\mu BC_2$ made earlier for arriving at the $SU_q(2)$ Poisson bracket structure is a special case of the procedure just outlined.

In this work we have shown that the classical spin dynamics in a constant uniform magnetic field can also be described by assuming that the generators of spin angular momentum satisfy $SU_q(2)$ Poisson bracket relations. Recently Hojman (1991) arrived at a similar conclusion by considering the closely related classical dynamics of a symmetric top within a Hamiltonian framework. Our analysis is conceptually simpler, formulated without recourse to a Hamiltonian theory of the equations of motion. Our analysis is inspired by a noteworthy article of Dyson (1990). In this paper Dyson assumes only the commutation relation between position and velocity of a spinless particle, and, introducing the magnetic and electric fields by way of definition he arrives at the Lorentz force equation in an operator form and derives the homogeneous equations of fields. We have extended Dyson's analysis to the present case of a charged particle spinning in a constant homogeneous magnetic field at the expense of introducing two functions denoted V and C. See (38) and (40). They are both constrained to be constants of motion. They can be quite arbitrary apart from being subject to the condition (41).

It is a pleasure to thank Dr Bani Mitra for having brought to my attention the interesting work of Hojman. This work is supported by the Council of Scientific and Industrial Research, New Delhi. I am grateful to Professor R Simon for local hospitality at the Institute of Mathematical Sciences during the workshop on Geometric Phases in Physics (7–11 October 1991).

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